DYNAMIC ONE-PILE NIM

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Introduction The purpose of this paper is to solve a class of combinatorial games consisting of one-pile counter pickup games for which the maximum number of counters that can be removed on each successive move changes during the play of the game. Two players alternate removing a positive number of counters from the pile. An ordered pair (N, x) of positive integers is called a *position*. The number N represents the size of the pile of counters and x represents the greatest number of counters that can be removed on the next move. A function $f : Z^+ \longrightarrow Z^+$ is given which determines the maximum size of the next move in terms of the current move size. Thus a *move* in a game is an ordered pair of positions $(N, x) \longmapsto (N - k, f(k))$, where $1 \le k \le \min\{N, x\}$. The game ends when there are no counters left, and the winner is the last player to move in the game. This paper extends two papers, one by Epp and Ferguson[2], and the other by Schwenk[6].

In order to introduce the concepts in this paper, we initially assume that f satisfies

$$(*) \qquad\qquad f(n+1) - f(n) \ge -1.$$

Later in the paper we prove the necessary and sufficient conditions on f so that our strategy is effective. In the appendix, we discuss the Epp, Ferguson paper. The authors are grateful to the referee for pointing out the possibility of finding both *necessary* and sufficient conditions on the function f so that the solution is effective.

The game of 'static' one-pile nim is well understood. These are called *subtraction games*. A pile of n counters and a constant k are given. Two players alternately take from 1 up to k counters from the pile. The winner is the last player to remove a counter. The theory of these games is complete. See [1, page 83].

Before discussing the strategy for playing dynamic one-pile nim, we prove four lemmas. These lemmas appear to have nothing in common with our games, but once they are proved, the strategy for playing will be easily understood.

Generalized Bases An infinite increasing sequence $B = (b_0 = 1, b_1, b_2, \ldots)$ of positive integers is called an *infinite g-base* if for each $k \ge 0$, $b_{k+1} \le 2b_k$. This 'slow growth' of B's members guarantees lemma 1. Finite g-bases. A finite increasing sequence $B = (b_0 = 1, b_1, b_2, \ldots b_t)$ of positive integers is called as *finite g-base* if for each $0 \le k < t$, $b_{k+1} \le 2b_k$.

Lemma 1. Let B be an infinite g-base. Then each positive integer N can be represented as $N = b_{i_1} + b_{i_2} + \cdots + b_{i_t}$ where $b_{i_1} < b_{i_2} < \cdots < b_{i_t}$ and each b_{i_j} belongs to B.

Proof. The proof is given by the following recursive algorithm. Note first that $b_0 = 1 \in B$. Suppose all the integers $1, 2, 3, \ldots, m-1$ have been represented as a sum of distinct members of B. Let b_k denote the largest element of B not exceeding m. That is, $b_k \le m < b_{k+1}$. Then $m = (m - b_k) + b_k$. Now $m - b_k < b_k$, for otherwise $2b_k \le m$.

But $b_{k+1} < 2b_k$, contradicting the definition of b_k. Since $m - b_k$ is less than m, it follows that $m - b_k$ has been represented as a sum of distinct members of B that are less than b_k. Thus we may suppose that $m - b_k = b_{i_1} + b_{i_2} + \cdots + b_{i_{t-1}}$ where $b_{i_1} < b_{i_2} < \cdots < b_{i_{t-1}}$ and each b_{i_j} belongs to B. Then $m = b_{i_1} + b_{i_2} + \cdots + b_{i_t}$, where $b_{i_t} = b_k$, $b_{i_1} < b_{i_2} < \cdots < b_{i_t}$ and each b_{i_j} belongs to B. ■

Note that in general it may be possible to represent an integer N as a sum of distinct members of B in more than one way. We now define a stable representation.

Definition Let $B = (b_0 = 1, b_1, \ldots)$ be an infinite g-base. Suppose $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k}$, where $b_{i_1} < b_{i_2} < \cdots < b_{i_k}$. We say that this representation of N is *stable* if for every t, $1 \le t \le k$,

$$\sum_{\theta=1}^{t} b_{i_\theta} < b_{i_t+1}.$$

Thus, in a stable representation of N, each member b_k of B is greater than the sum of all the summands b_{i_k} of N that are less than b_k.

Lemma 2. Let $B = (b_0 = 1, b_1, \ldots)$ be an infinite g-base. Then each positive integer N has exactly one stable representation. It is generated by the algorithm used in the proof of Lemma 1.

Proof. Let us first suppose that $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k}$, where $b_{i_1} < b_{i_2} < \cdots < b_{i_k}$ is a stable representation of N. We show that this representation is unique and is generated by the algorithm of Lemma 1. The proof is by mathematical induction on N. For $N = 1$, the representation is certainly unique and generated by the algorithm. Next we show that b_{i_k} is uniquely generated by the algorithm. Let $b_s \le N < b_{s+1}$. Then $b_{i_k} \le N < b_{s+1}$. If $b_{i_k} < b_s$, then $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k} < b_{i_k+1} \le b_s$, contradicting the assumption that $b_s \le N < b_{s+1}$. Therefore, $b_{i_k} \in B, b_{i_k} \ge b_s$, and $b_{i_k} < b_{s+1}$ which together imply that $b_{i_k} = b_s$. This means that b_{i_k} is unique and is computed by the algorithm. Now since $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k}$ is a stable representation of N, it follows from the definition of stable representation that $N - b_{i_k} = b_{i_1} + b_{i_2} + \cdots + b_{i_{k-1}}$ is a stable representation of $N - b_{i_k}$. Therefore, by induction we see that each of $b_{i_1}, b_{i_2}, \ldots, b_{i_{k-1}}$ is also unique and generated by the algorithm. We next show that any number N has at least one stable representation. To do this, let $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k}$, where $b_{i_1} < b_{i_2} < \cdots < b_{i_k}$, be generated by the algorithm. We prove by induction on N that this representation is stable. Again the case $N = 1$ is trivial. Suppose $b_s \le N < b_{s+1}$. Then by definition of the algorithm, $b_{i_k} = b_s$ and

$$N = \sum_{\theta=1}^{k} b_{i_\theta} < b_{s+1} = b_{i_k+1}.$$

Note that $N - b_{i_k} = b_{i_1} + b_{i_2} + \cdots + b_{i_{k-1}}$. Also, by definition of the algorithm, we see that each of $b_{i_1}, b_{i_2}, \ldots, b_{i_{k-1}}$ is generated by the algorithm using the number $N - b_{i_k}$. Therefore, by induction on $N - b_{i_k}$, we know that $b_{i_1} + b_{i_2} + \cdots + b_{i_{k-1}}$ is a stable representation of $N - b_{i_k}$. Therefore, by the definition of stable representation, we know that for every $1 \leq t \leq k-1$,

$$\sum_{\theta=1}^{t} b_{i_\theta} < b_{i_t+1}.$$

Therefore, for every $1 \leq t \leq k$,

$$\sum_{\theta=1}^{t} b_{i_\theta} < b_{i_t+1}. \qquad \blacksquare$$

<u>*Generating g-bases*</u> For every function $f : Z^+ \to Z^+$ satisfying

$$(*) \qquad\qquad f(n+1) - f(n) \geq -1,$$

we generate a g-base B_f as follows:
Let $b_0 = 1$. Suppose $(b_0, b_1, \ldots, b_k)$ have been generated. Then $b_{k+1} = b_k + b_i$, where b_i is the smallest member of $\{b_0, b_1, \ldots, b_k\}$ such that $f(b_i) \geq b_k$, if such a b_i exists. If no such b_i exists for some k, the base B_f is finite. In this part of the paper we assume that B_f is infinite. As an example, if $f(n) = 2n$, then $B_f = \{1, 2, 3, 5, 8 \ldots\}$ and we have what is called *Fibonacci Nim*.

For lemmas 3 and 4 we assume that $B_f = (b_0 = 1, b_1, \ldots)$ is the infinite g-base generated by a function f satisfying the inequality $(*)$, and that the positive integer N has stable representation $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k}$ with $b_{i_1} < b_{i_2} < \cdots < b_{i_k}$.

<u>*Lemma 3.*</u> $f(b_{i_1}) < b_{i_2}$.

<u>*Proof.*</u> Because the representation is stable, $b_{i_1} + b_{i_2} < b_{i_2+1} \leq b_{i_3}$. Now $b_{i_2+1} = b_{i_2} + b_i$ where b_i is the smallest member of $b_0, b_1, \ldots, b_{i_2}$ such that $f(b_i) \geq b_{i_2}$. Since $b_{i_2} + b_{i_1} < b_{i_2+1}$, it follows that b_i is larger than b_{i_1}. Since b_i is the smallest member of $\{b_0, b_1, \ldots, b_{i_2}\}$ such that $f(b_i) \geq b_{i_2}$, it follows that $f(b_{i_1}) < b_{i_2}$. $\blacksquare$

<u>*Lemma 4.*</u> Suppose integer x satisfies $1 \leq x < b_{i_1}$. Let $b_{i_1} - x = b_{j_1} + b_{j_2} + \cdots + b_{j_t}$ be the stable representation of $b_{i_1} - x$ in B_f, where $b_{j_1} < b_{j_2} < \cdots < b_{j_t}$. Then

(1) $N - x = b_{j_1} + b_{j_2} + \cdots + b_{j_t} + b_{i_2} + b_{i_3} + \cdots + b_{i_k}$ is the stable representation of $N - x$ in B_f and

(2) $b_{j_1} \leq f(x)$.

<u>*Proof.*</u> The proof of (1) is trivial. The proof of (2) is by mathematical induction on t. We consider below two cases, the first of which takes care of $t = 1$.

Case (a): $1 \leq b_{i_1} - x \leq b_{i_1-1}$.

Case (b): $b_{i_1-1} < b_{i_1} - x < b_{i_1}$.

In case (a), we show that $f(x) \geq b_{i_1} - x$. Since $b_{i_1} - x = b_{j_1} + b_{j_2} + \cdots + b_{j_t}$, it follows that $f(x) \geq b_{j_1}$. Now $b_{i_1} = b_{i_1-1} + b_i$, where b_i is the smallest member of $b_0, b_1, \ldots, b_{i_1-1}$ such that $f(b_i) \geq b_{i_1-1}$. Therefore, $f(b_i) = f(b_{i_1} - b_{i_1-1}) \geq b_{i_1-1}$. Note that the condition $f(n+1) - f(n) \geq -1$ can be used repeatedly to see that $f(n+N) - f(n) \geq -N$. Thus $f(n+N) \geq f(n) - N$ and

$$
\begin{aligned}
f(x) &= f\left(b_{i_1} - b_{i_1-1} + [b_{i_1-1} - (b_{i_1} - x)]\right) \\
&\geq f(b_{i_1} - b_{i_1-1}) - [b_{i_1-1} - (b_{i_1} - x)] \\
&= f(b_i) + b_{i_1} - b_{i_1-1} - x \\
&\geq b_{i_1-1} + b_{i_1} - b_{i_1-1} - x = b_{i_1} - x,
\end{aligned}
$$

since $f(b_i) \geq b_{i_1-1}$. That is, $f(x) \geq b_{i_1} - x$. Note that case (a) completely takes care of the lemma when $t = 1$ and starts the mathematical induction on t.
Case (b) Since $b_{i_1} - x = b_{j_1} + b_{j_2} + \cdots + b_{j_t}$ is stable where $b_{j_1} < b_{j_2} < \cdots < b_{j_t}$ and since $b_{i_1-1} < b_{i_1} - x = b_{j_1} + b_{j_2} + \cdots + b_{j_t} < b_{i_1}$, we know from lemma 2 (or directly from the definition of stable itself) that $b_{j_t} = b_{i_1-1}$. Therefore,

$$
\begin{aligned}
x &= b_{i_1} - (b_{j_1} + b_{j_2} + \cdots + b_{j_t}) \\
&= (b_{i_1} - b_{i_1-1}) - (b_{j_1} + b_{j_2} + \cdots + b_{j_{t-1}})
\end{aligned}
$$

Now $b_{i_1} = b_{i_1-1} + b_i$ where b_i is the smallest member of $b_0, b_1, \ldots, b_{i_1-1}$ such that $f(b_i) \geq b_{i_1-1}$. Therefore $x = b_i - (b_{j_1} + b_{j_2} + \cdots + b_{j_{t-1}})$; that is, $b_{j_1} + b_{j_2} + \cdots + b_{j_{t-1}} = b_i - x$. Of course, $b_{j_1} + b_{j_2} + \cdots + b_{j_{t-1}}$ is stable. Therefore, by mathematical induction, $f(x) \geq b_{j_1}$. ■

Theorem 1 puts these four lemmas together to establish a strategy for playing dynamic one-pile nim optimally when B_f is infinite.
Theorem 1. Suppose the dynamic one-pile nim game with initial position (N, x) and move function f satisfying $(*)$ is given, and the g-base B_f is infinite. Also, let $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k}$ be the stable representation of N in B_f, where $b_{i_1} < b_{i_2} < \cdots < b_{i_k}$. Then the first player can win if $x \geq b_{i_1}$ and the second player can win if $x < b_{i_1}$.

Proof. Assuming $x \geq b_{i_1}$, the first player removes b_{i_1} counters. This move results in the position $(N - b_{i_1}, f(b_{i_1})) = (b_{i_2} + b_{i_3} + \cdots + b_{i_k}, f(b_{i_1}))$. Note that the number of summands in the stable representation of the pile size N of the position has been reduced. Also, the representation of $N - b_{i_1}$ is stable and, by lemma 3, $f(b_{i_1}) < b_{i_2}$.

Thus the second player must remove fewer than b_{i_2} counters. Suppose the second player removes x' counters, where $1 \leq x' < b_{i_2}$. Thus the second player has moved to a position $(N - b_{i_1} - x', f(x')) = (b_{i_2} + b_{i_3} + \cdots + b_{i_k} - x', f(x')) = (b_{j_1} + b_{j_2} + \cdots + b_{j_t} + b_{i_3} + \cdots + b_{i_k}, f(x'))$, where $b_{j_1} + b_{j_2} + \cdots + b_{j_t}$ is the stable representation of $b_{i_2} - x'$. By lemma 4, parts 1 and 2, $b_{j_1} + b_{j_2} + \cdots + b_{j_t} + b_{i_3} + \cdots + b_{i_k}$ is the stable representation of $b_{i_2} + b_{i_3} + \cdots + b_{i_k} - x'$ and $b_{j_1} \leq f(x')$.

Note that the second player has not reduced the number of summands, and after his move, $b_{j_1} \leq f(x')$. The first player is therefore in a position analogous to the initial position, since $b_{j_1} \leq f(x')$. The first player can now reduce the pile by b_{j_1} counters, which again reduces the number of summands. Thus the first player can reduce the number of summands and the second player cannot. This means that the first player will eventually reduce the number of summands to zero, thereby winning.

When the initial position satisfies $x < b_{i_1}$, the second player wins by using the first player's strategy in the case above, that is, by reducing the pile size by the smallest number b_{i_1} that appear in the stable representation of the pile size. ■

Next we discuss the case in which the g-base B_f is finite. Note that when f is bounded, B_f is finite. However, a finite g-base is possible even when f is unbounded. As an example consider $f : Z^+ \to Z^+$ defined by $f(1) = f(2) = f(3) = 2$ and $f(n) = n$ for all $n \geq 4$. This function satisfies the unit jump condition $f(n+1) - f(n) \geq -1$. Its g-base is $b_0 = 1, b_1 = b_0 + b_0 = 2$, $b_2 = b_1 + b_0 = 3$. Of course, b_3 does not exist because there is no member $b_i \in \{b_0, b_1, b_2\} = \{1, 2, 3\}$ such that $f(b_i) \geq b_2 = 3$. Thus the g-base is finite. The proofs of the following four lemmas and the theorem parallel very closely the proofs of the corresponding four lemmas and the theorem for infinite g-bases.

Lemma 1′. Let $B = (b_0 = 1, b_1, b_2, \ldots, b_t)$ be a finite g-base. Then each positive integer N can be represented as a sum of distinct members of B allowing multiple copies of the largest element of B:

$$N = b_{i_1} + b_{i_2} + \cdots + b_{i_k} + \theta b_t,$$

where $b_{i_1} < b_{i_2} < \cdots < b_{i_k} < b_t$ for some integer $\theta \geq 0$.

As we noted in the case for infinite g-bases, there may be multiple representations. Thus we have the following definition of stable representation.

Definition Let $B = (b_0 = 1, b_1, \ldots, b_t)$ be a finite g-base. Suppose $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k} + \theta b_t$, where $b_{i_1} < b_{i_2} < \cdots < b_{i_k} < b_t$ and θ is a nonnegative integer. We say that this representation of N is *stable* if for every h, $1 \leq h \leq k$,

$$\sum_{\phi=1}^{h} b_{i_\phi} < b_{i_h+1}.$$

<u>*Lemma 2′.*</u> Let $B = (b_0 = 1, b_1, \ldots, b_t)$ be a finite g-base. Then each positive integer N has exactly one stable representation.

For lemmas 3′ and 4′ we assume that $B_f = (b_0 = 1, b_1, \ldots, b_t)$ is the finite g-base generated by a function $f : Z^+ \longrightarrow Z^+$ satisfying the inequality $(*)$, and that the positive integer N has stable representation $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k} + \theta b_t$ with $b_{i_1} < b_{i_2} < \cdots < b_{i_k} < b_t$ and θ is a nonnegative integer.

<u>*Lemma 3′.*</u> $f(b_{i_1}) < b_{i_2}$.

Note that for all $b_i \in B_f$, $f(b_i) < b_t$. This is why B_f is finite.

<u>*Lemma 4′.*</u> Suppose integer x satisfies $1 \leq x < b_{i_1}$. Let $b_{i_1} - x = b_{j_1} + b_{j_2} + \cdots + b_{j_h}$ be the stable representation in B_f, where $b_{j_1} < b_{j_2} < \cdots < b_{j_h}$. Then

1. $N - x = b_{j_1} + b_{j_2} + \cdots + b_{j_h} + b_{i_2} + b_{i_3} + \cdots + b_{i_k} + \theta b_t$ is the stable representation of $N - x$ in B_f and

2. $b_{j_1} \leq f(x)$.

<u>*Theorem 1′.*</u> Suppose the dynamic one-pile nim game with initial position (N, x) and move function f satisfying $(*)$ is given, and the g-base $B_f = (b_0 = 1, b_1, b_2, \ldots, b_t)$ is finite. Also, let $N = b_{i_1} + b_{i_2} + \cdots + b_{i_k} + \theta b_t$ be the stable representation of N in B_f, where $b_{i_1} < b_{i_2} < \cdots < b_{i_k} < b_t$. Then the first player can win if $x \geq b_{i_1}$ and the second player can win if $x < b_{i_1}$. In the special case where $N = \theta b_t$, the first player can win if $x \geq b_t$, and the second player can win if $x < b_t$.

We now turn our attention to the converse problem. Let $f : Z^+ \to Z^+$ be any function. We find necessary and sufficient conditions on f so that theorem 1 is true. We generate a g-base $B_f = (b_0 = 1, b_1, \ldots)$ from f just as before. For convenience, we assume B_f is infinite. Lemmas 1-3 remain true since they do not depend on the condition

$$(*) \qquad\qquad f(n+1) - f(n) \geq -1.$$

Also, in the proof of lemma 4, only case (a) of property (2) used property $*$ on f.

Definition. For any positive integer N, let $N = b_{i_1} + b_{i_2} + \ldots + b_{i_k}$ be the stable representation of N in B_f, where $b_{i_1} < b_{i_2} < \ldots < b_{i_k}$. Then we define $g(N) = b_{i_1}$. Also, $g(0) = g(-N) = 0$.

Lemma 5. Given $f : Z^+ \to Z^+$, theorem 1 is true for f if and only if lemma 4 is true for f.

Proof. Obviously lemma 4 implies theorem 1. We now show that if lemma 4 is false, then theorem 1 is false. Since part (1) of lemma 4 is trivial, we can use the definition of g to see that lemma 4 is equivalent to the statement for all $b_\theta \in B_f$, and for all $1 \leq x < b_\theta$,

$$g(b_\theta - x) \leq f(x).$$

No matter what f is, $b_0 = 1, b_1 = 2, g(1) = 1, g(2) = 2$ holds. Therefore, $g(b_\theta - x) \leq f(x)$ holds when $b_\theta \in \{b_0, b_1\}$ and $1 \leq x < b_\theta$ for all f. Define b_ϕ to be the smallest member of $\{b_2, b_3, b_4, \ldots\}$ such that $g(b_\phi - x) > f(x)$ for some $1 \leq x < b_\phi$. By definition of b_ϕ, this means that lemma 4 is true for all $b_\theta \in \{b_0, b_1, \ldots, b_{\phi-1}\}$ and all $1 \leq x < b_\theta$. This means that theorem 1 holds for all positions (N, x) when $1 \leq N < b_\phi$ since the base members $b_\phi, b_{\phi+1}, b_{\phi+2}, \ldots$ do not come into play when $N < b_\phi$. Next consider the position (b_ϕ, x) as described above. Of course, $1 \leq x < b_\phi$ and $g(b_\phi - x) > f(x)$. We will show that (b_ϕ, x) is an unsafe position, which contradicts theorem 1. Let the first player remove x counters so that $(b_\phi, x) \mapsto (b_\phi - x, f(x))$. Since $b_\phi - x < b_\phi$, theorem 1 correctly tells us whether $(b_\phi - x, f(x))$ is safe or unsafe. Because $f(x) < g(b_\phi - x)$, theorem 1, along with the definition of g tells us that $(b_\phi - x, f(x))$ is a safe position. This means that (b_ϕ, x) is an unsafe position. ■

Lemma 6. The necessary and sufficient conditions on f so that lemma 4 holds is that for all $b_{i_1} \in \{b_1, b_2, \ldots\}$, and for all $1 \leq b_{i_1} - x \leq b_{i_1 - 1}$, $\quad g(b_{i_1} - x) \leq f(x)$. *Proof.* First note that part (1) of lemma 4 is a trivial statement and can be ignored. So what we are saying here is that lemma 4 is true if and only if lemma 4 is true for part (2), case (a). Note in part (2) that $b_{j_1} = g(b_{i_1} - x)$, from the definition of g, since $b_{i_1} - x = b_{j_1} + b_{j_2} + \cdots + b_{j_t}$ is the stable representation of $b_{i_1} - x$ in B_f and $b_{j_1} < b_{j_2} < \ldots b_{j_t}$.

The reason lemma 4 is true if and only if lemma 4 is true for part (2), case (a) is that the only place in the proof of lemma 4 where the property $*$ is used is in proving part (2), case (a). Since we have dropped the condition $*$ on f, the only way that we can now deal with part (2), case (a) is just to assume that lemma 4 is always true for part (2), case (a). Thus part (2), case (a) becomes the necessary and sufficient condition on f for lemma 4 to hold. ■

Definition. For all nonnegative integers k, let

$$b_{\theta(k)} = b_{k+1} - b_k,$$

where $b_{\theta(k)} \in \{b_0, b_1, b_2, \ldots, b_k\}$.

<u>*Lemma 7.*</u> The following two conditions are equivalent.

1. For all $b_{k+1} \in \{b_1, b_2, \ldots\}$ and for all $1 \leq b_{k+1} - x \leq b_k$,

$$g(b_{k+1} - x) \leq f(x).$$

2. For all nonnegative integers k and for all nonnegative integers $\overline{x}$,

$$g(b_k - \overline{x}) \leq f(b_{\theta(k)} + \overline{x}).$$

Note that (1) is a restatement of the condition in lemma 6. Also, (2) uses $g(0) = g(-N) = 0$.

<u>*Proof.*</u> We first show that (1) implies (2). Since $g(0) = g(-N) = 0$, let us assume $1 \leq b_k - \overline{x}$. Let $x = b_{\theta(k)} + \overline{x}$. Thus, $x = b_{k+1} - b_k + \overline{x}$. Therefore $1 \leq b_{k+1} - x = b_k - \overline{x} \leq b_k$. Hence from (1), $g(b_{k+1} - x) = g(b_k - \overline{x}) \leq f(x) = f(b_{\theta(k)} + \overline{x})$. That is $g(b_k - \overline{x}) \leq f(b_{\theta(k)} + \overline{x})$. We now show that (2) implies (1). Since $b_{k+1} - x \leq b_k$, define $\overline{x}$ by $b_{k+1} - x + \overline{x} = b_k$, where $\overline{x} \geq 0$. Therefore, $b_k - \overline{x} = b_{k+1} - x$. Also, $x = b_{k+1} - b_k + \overline{x} = b_{\theta(k)} + \overline{x}$. Therefore from (2), $g(b_k - \overline{x}) = g(b_{k+1} - x) \leq f(b_{\theta(k)} + \overline{x}) = f(x)$. That is, $g(b_{k+1} - x) \leq f(x)$. ■

<u>*Main Theorem.*</u> Given $f : Z^+ \to Z^+$ with an infinite B_f, the necessary and sufficient conditions on f so that theorem 1 holds for f is that for all nonnegative k and $\overline{x}$

$$g(b_k - \overline{x}) \leq f(b_{\theta(k)} + \overline{x}).$$

Since $g(N) \leq N$ observe that the following are sufficient but not necessary conditions on f for theorem 1 to hold: for all nonnegative integers k and $\overline{x}$,
$f(b_{\theta(k)} + \overline{x}) \geq b_k - \overline{x}$. Recall that $f(b_{\theta(k)}) \geq b_k$ from the definition of B_f. From this it is easy to see that the original restriction $(*)$ on f implies $f(b_{\theta(k)} + \overline{x}) \geq b_k - \overline{x}$.

The following theorem allows the main theorem to be used more efficiently since we only have to worry about $f(x)$ when x is not in the base B_f.

<u>*Theorem 2.*</u> Suppose that $f : Z^+ \to Z^+$ generates the infinite g-base $B_f = \{b_0 = 1, b_1, b_2, \ldots\}$, and f is non-decreasing on B_f. Then f satisfies the hypothesis of the main theorem if and only if the following is true for each x not in B_f. Suppose $b_t < x < b_{t+1}$. Also, suppose $b_{\theta(k)} < x < b_{k+1}$ if and only if $k \in \{t, t+1, t+2, \ldots, t+\overline{t}\}$. Then for this x, we require $g(b_{t+i} - x) \leq f(x)$ for $i = 1, 2, 3, \ldots \overline{t} + 1$.

The proof of this, which uses part 1 of lemma 7 is left to the reader. Using this theorem, we see that f generates the Fibonacci base $B_f = \{1, 2, 3, 5, 8, 13, \ldots\}$ and the main theorem is effective for f if and only if the following two conditions hold: a. for every $b_t \in B_f, b_{t+1} \leq f(b_t) < b_{t+2}$ and b. for all nonnegative integers t, and all x satisfying $b_t < x < b_{t+1}$, $g(b_{t+1} - x) \leq f(x)$. Note the $g(b_{t+1} - x) = g(b_{t+2} - x)$ when $b_t < x < b_{t+1}$, so $g(b_{t+2} - x) \leq f(x)$ is redundant.

The misère version To win at the misère version (N, x) of dynamic nim, simply use the theory to win the game $(N-1, x)$, so that your opponent is forced to take the last counter.

Appendix We now discuss theorem 2.1 of the Epp Ferguson paper. Let $f : Z^+ \to Z^+$ be an arbitrary function defining our one pile dynamic nim game. If a player is confronted with a pile size of $n \geq 1$, let $L(n)$ denote the smallest possible winning move. Of course, $L(n) \leq n$ and equality might hold. Note also that removing k counters from a pile of n is a winning move if and only if $f(k) < L(n-k)$, where $L(0) = \infty$. Theorem 2.1 (Epp, Ferguson): Suppose $f(k) < L(n-k)$. Then $k = L(n)$ if and only if $L(k) = k$. Epp and Ferguson prove this when f is non-decreasing. The reader can easily show that if f satisfies the condition of our main theorem, then $L(L(n)) = L(n)$ for all positive integers n. The following example shows that Theorem 2.1 breaks down when f is not non-decreasing.

Example. There exists f satisfying $f(n+1) - f(n) \geq -1$ such that there exists $k < n$ with $f(k) < L(n-k)$, $L(k) = k$, and $k \neq L(n)$. *Proof.* Consider f defined by $f(n) = 8 - n$ when $1 \leq n \leq 7$ and $f(n) = n$ when $8 \leq n$. Then $B_f = \{1, 2, 3, 4, 5, 6, 7, 8, 16, 32, 64, 128, 256, \ldots\}$. Since $9 = 8 + 1$, we see that $L(9) = 1$. Consider the position $(9, 8)$. We see that the following are all winning moves:

$(9,8) \mapsto (9-7, f(7)) = (2,1), L(7) = 7 \neq L(9) = 1,$

$(9,8) \mapsto (9-6, f(6)) = (3,2), L(6) = 6 \neq L(9),$

$\vdots$

$(9,8) \mapsto (9-2, f(2)) = (7,6), L(2) = 2 \neq L(9).$ ■

The reader might like to show that for the following f, $L(16) = 10$, and $L(10) \neq 10$: $f(n) = n, n \neq 10$, and $f(10) = 1$. Of course this f does not satisfy the conditions of our main theorem.

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